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Finding a Cycle in a Graph with Minimum Cost to Time Ratio

with Application to a Ship Routing Problem

by

G. B. Dantzig*, W. O. Blattner**, and M. R. Rao**

Associated with each arc (i,j) of a graph are two numbers c_{ij} the "cost" and t_{ij} the "time" per unit flow. In our application the unit flow is a ship making one trip from i to j at a cost c_{ij} and taking t_{ij} hours. In another example, a vessel for hire can make a profit p_{ij} each time it goes from i to j; eventually (if there are a finite number of ports) it must complete a cycle with a total profit P and a total lapsed time T where P is the sum of the profits and T is the sum of the times on the arcs of the cycle. For a maximum rate of return, the shipowners should use that cycle which maximizes the ratio of P/T. Later we shall describe a more complex linear programming model which we solve using a column generation scheme (a variant of the decomposition principle). The subproblem turns out to be one of finding a cycle in a graph that has the minimum ratio of total cost to total time.

Consider the following linear program:

Find Min z, $x_{ij} \geq 0$ such that

$$(1) \quad \sum_{i,j=1}^{n} c_{ij}\, x_{ij} = z$$

$$(2) \quad \sum_{i,j=1}^{n} t_{ij}\, x_{ij} = 1 \qquad\qquad t_{ij} \geq 0$$

$$(3) \quad \sum_{i=1}^{n} x_{ij} - \sum_{k=1}^{n} x_{jk} = 0 \qquad\qquad j = 1,2,\ldots,n$$

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Theorem 1: Associated with an extreme minimizing solution to (1), (2), (3) is a cycle whose total cost to time ratio is minimum.

Proof: Let $\bar{x}_{ij} = 1$ if (i,j) is an arc of some cycle and $\bar{x}_{ij} = 0$ otherwise. Let $\Sigma\, t_{ij}\, \bar{x}_{ij} = T$, then $x_{ij} = \bar{x}_{ij}/T$ satisfies (2) and (3) and $z = C/T$ is the ratio of total costs, $C = \Sigma\, \Sigma\, c_{ij}\, \bar{x}_{ij}$, to total time T.. Accordingly we can always associate with a cycle one of the solutions of (1), (2), (3).

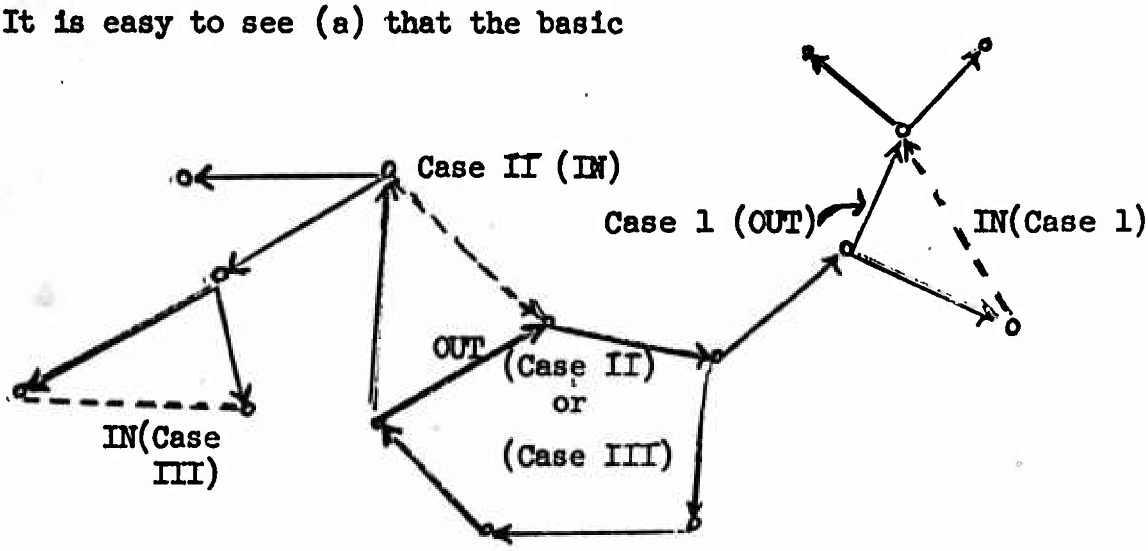
Consider now the class of minimizing solutions to (1), (2), (3). We can now see that to an extreme minimizing solution corresponds a simple cycle. This follows because the flows $x_{ij} \geq 0$ can be represented as a sum of simple circulations. If any of these circulations had by itself a lower ratio $\Sigma c_{ij}x_{ij}/\Sigma t_{ij}x_{ij}$ than another one, the solution could not be optimal. Indeed an improved solution could be obtained by building up the circulation around that cycle with the lowest ratio and decreasing the flow around the one with a higher ratio. Nor could a solution be extreme if there were two simple cycles with the same ratio because one could represent such a solution as a convex combination of two others by first building up and then building down the circulation in one of the cycles while adjusting the other so (2) holds.

THE SIMPLEX ALGORITHM

A simple algorithm for solving (1), (2), (3) can be derived from the simplex method. A basis involves n columns (one equation is redundant). The corresponding arcs in the graph must consist of a tree and one out-of-tree arc. To see this we note that since a basis is non-singular, there must be at least one non-singular $(n-1)x(n-1)$ submatrix formed by deleting the row associated with the time equation and deleting some column of the basis. Non-singularity implies that the $n-1$ arcs associated with the remaining columns form a tree. The arc associated with the variable of the deleted column together with a subset of the arcs of the tree form a cycle. In fact, this implies that <u>every</u> basic feasible solution to (2) and (3) must consist of one tree spanning all nodes and one <u>simple</u> cycle formed by a subset of the arcs of the tree augmented by one additional arc. This additional arc completes the cycle making possible the positive flow forced by the time equation.

It is easy to see (a) that the basic

(4)



variables other than the cycle variables have zero value in a basic solution, (b) that each node in the simple cycle has one cycle arc pointing into and the other away from it, and (c) the values of the cycle variables are the same and equal to $1/T$ where T is the total time around the cycle.

It is also easy to compute the simplex multipliers (prices) associated with the basis. Indeed if we let ρ be the the multiplier associated with the time equation (2) and let Π_j be those associated with the node equations (3), then for each arc (i,j) associated with a basic variable

$$\Pi_j - \Pi_i + \rho\, t_{ij} = c_{ij}$$

Summing these relations for all arcs (i,j) ϵ Cycle yields

(5) $$\rho = C/T = \Sigma\, c_{ij} / \Sigma\, t_{ij} \qquad (i,j) \in \text{cycle arcs}$$

Knowing ρ , one may arbitrarily choose the value of any one node (there is a redundant equation) and determine the remainder by

(6) $$\Pi_j = \Pi_i + (c_{ij} - \rho\, t_{ij}) \qquad (i,j) \in \text{tree arcs}$$

by branching out from the selected node along arcs (i,j) of the tree until all nodes are reached.

To obtain an improved solution the simplex multipliers are used to eliminate the basic variables from the cost equation. The resulting coefficients for the non-basic variables are

(7) $$\delta_{ij} = (c_{ij} - \rho\, t_{ij}) - (\Pi_j - \Pi_i)$$

If all $\delta_{ij} \geq 0$, then the value of ρ given by (5) is the minimum cost to time ratio and the problem is solved.

If not, let

(8) $$\delta_{pq} < 0 \qquad \text{for some } (p,q)$$

We now make a special Inductive Assumption: at each iteration, there is a basic feasible solution consisting of a directed tree spanning out from a single node that is its root, augmented by one additional arc to form one simple cycle.†

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† If a feasible cycle exists in the graph we can satisfy the inductive assumption by taking any node of the cycle as the root of the tree and spanning out from the nodes of the cycle using forward arcs to other nodes, and then iteratively, repeating the process with all nodes reached. If nodes still remain they can be reached by introducing high cost artificial arcs as required. If there is no feasible cycle in the graph there is obviously no feasible solution to the problem; if such is indeed the case this fact will be discovered by the algorithm that follows.

If (p,q) is the dotted arc marked "IN (Case I)" in the figure, then it is easy to see that entering (p,q) into the set of basic arcs does not form a new cycle and we must drop out of the basic set the one which is also directed into q. A new basic feasible set of arcs is obtained with the values of Π_i decreased to $\Pi_i + \delta_{pq}$ for node $i = q$ and all nodes i that are followers of q in the new tree. All other Π_i remain unchanged. In Cases II and III a new cycle is formed and we must drop out of the basis any one arc (r,s) which is in the old cycle and not in the new cycle.

Theorem 2: If the inductive assumption holds, and if (p,q) is entered into the basic set in place of the basic arc directed into q, then the inductive assumption holds after the change except when a new cycle is formed.

Theorem 3: If a new cycle is obtained as a result of changes in the basic set of arcs, its $\rho^* = C/T$ ratio is less than the previous one.

Theorem 4: If $d_{ij} = (c_{ij} - \rho\, t_{ij})$ is used as distances on arcs (i,j) in the graph, then any cycle in the graph whose sum of distances around the arcs of the cycle is negative has a lower C/T ratio.

The simplex method accordingly reduces down to finding a negative cycle in a graph when arc distances d_{ij} given: Starting with $\Pi_i = 0$ for some node of the cycle, the other Π_i are simply the distances from this origin node along arcs of the tree to node i. The simplex iterative process is seen to be the standard one for determining the shortest route from the origin to all others, terminating when either

(9) $$d_{ij} - (\Pi_j - \Pi_i) \geq 0 \qquad \text{for all } i,j$$

or on some iteration a negative cycle like in Cases II or II is found. In the former case, it is seen by summing (9) for all (i,j) around any given cycle that distance around any cycle is non-negative so that the ρ used to determine $d_{ij} = c_{ij} - \rho\, t_{ij}$ is the best ratio. In the latter case, a new cycle with a lower value of ρ is obtained and new d_{ij} values are computed.

ALGORITHM: In the following algorithm the pairs (i,j) represent directed arcs defined in the graph:

0: Let S_0 be any starting cycle. If none available, set $\rho_{\bullet} = \text{Max } [c_{ij}/t_{ij}]$ in Step 2 below

1: For $k = 0,1,\ldots$

2: Compute $\rho_k = \Sigma\, c_{ij}/\Sigma t_{ij} \qquad (i,j) \in S_k$

3: Compute $d^k_{ij} = c_{ij} - \rho_k t_{ij} \qquad$ for all i, j

4: Set $\Pi^k_1 = 0$ and set predecessor of node 1 as * (meaning none).

Set $\Pi^k_i = \infty$ and set predecessor for $i \neq 1$ as 1.††

5: For each $i = 1,2,\ldots,n$ form $\delta^k_{ij} = d^k_{ij} + \Pi^k_i - \Pi^k_j$ for $j = 1,2,\ldots,n \quad j \neq i$.

(a) If $\delta^k_{ij} \geq 0$ return to (5) and continue scanning j for fixed i and then repeat increasing i to $i + 1$ until $i = n$, $j = n - 1$. If $i = n$, $j = n - 1$ terminate. Cycle S_k is optimal.

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††These are devices to initiate the computation without effort and to construct the starting directed tree necessary to satisfy the inductive assumption.

(b) If $\delta_{ij}^{k} < 0$ go to (6).

6: Determine the nodes in reverse order along the route R from the origin to i by back tracing the predecessors of i.

(a) If j is not a predecessor of i, then change predecessor of j to i and replace value of Π_j^k by $\Pi_j^k + \delta_{ij}^k$, return to (5) with $i = 1$, $j = 2$.

(b) If j is a predecessor of i, then let S_{k+1} be the cycle along the route R from node j to i and back to j along arc (i,j). Return to 1 increasing k to $k + 1$.

For programming simplicity the above algorithm does not maintain a directed tree. If it is modified to do so, the nodes can be priced sequentially along the tree and the return from step 6 (a) to step (5) modified to take advantage of this.

COMPUTATIONAL EXPERIENCE

Set I

Problem	Nodes	Arcs	Seconds on IBM 1620 (including input-output)
A	4	12	3.60
B	4	12	2.16
C	5	20	3.96
D	4	7	2.52
E	6	13	6.84

Set II

Problem	Nodes	Arcs	(Excluding input-output)
F	5	20	1.80
	10	90	7.92
H	15	210	14.04
I	20	380	33.84
J	25	600	36.00
K	30	870	103.68

For problems in Set II, the t_{ij} values for each arc were randomly generated integers between 10 and 60. Similarly, the c_{ij} values were randomly generated between 20 and 120.

We do not have an upper bound on the number of operations except the kind that one could derive from a standard proof of the simplex algorithm. In another paper where a variant of the scanning procedure given here is used an upper bound of (Nodes + 3) (Arcs) additions-comparison operations is given for finding a negative cycle, [1].

Application to a Ship Routing Problem

Amounts b_{ij} are required to be shipped from ports i to ports j. There are n ports (nodes). The shipping can either be done by charter at a cost v_{ij} per unit shipped or by using one of a fleet of m vessels under the control of the shipping company. If vessel k is used the amount that it carries between (i,j) depends on the kind of ship and on the pattern of ports forming a cycle g that is assigned to the ship. We denote this by a^{ij}_{kg}. Thus if arc (i,j) is not part of cycle g, then $a^{ij}_{kg} = 0$ and if it is its value is the capacity w^k_{ij} of the vessel. †††

Material balance equations: For $i, j = 1, 2, \ldots, n$

$$(10) \qquad y_{ij} + \sum_{k=1}^{m} \sum_{g} a^{ij}_{kg} \, x_{kg} = b_{ij}$$

where y_{ij} is amount chartered and x_{kg} is the number of times that ship k is employed in the g-th type cycle. We allow x_{kg} to have fractional values which we interpret as rate of use of the ship in some given period of time.

Vessel hour constraints:

$$(11) \qquad \sum_{g} t_{kg} \, x_{kg} + s_k = h_k \qquad k = 1,2,\ldots,m.$$

where h_k is the total hours available on the k-th vessel, s_k is the unused hours of the ship, t_{kg} is the time to complete one cycle of type g.

Objective to be minimized:

$$(12) \qquad \sum_{(i,j)} v_{ij} \, y_{ij} - \sum_{k} c_k \, s_k = z$$

††† Dependence on (i,j) is possible if type of cargo on route (i,j) is different from that on other arcs. In case of airplanes, capacity depends on distance.

Here we are assuming that the cost to operate vessel is c_k per unit time used, hence there is a savings of c_k per hour not used.

In an ore shipment application which we were interested in there were too many possible cycles to explicitly list all the coefficients of the problem. Accordingly we decided to generate the column of coefficients as needed. Using y_{ij} and s_k as basic variables, one has a starting basic feasible solution. We now assume we have introduced into the basis several other columns and we have a set of simplex multipliers p_{ij} associated with (10) and q_k with (11). We wish to "price out" the column associated with x_{kg} and to find that column g for each k that prices out most negative. The relative cost coefficient of x_{kg} becomes

$$(13) \qquad -q_k \; t_{kg} - \sum_{(i,j)} p_{ij} \; a_{kg}^{ij} = \sum_{(i,j)\epsilon g} (-q_k \; t_{ij}^k - p_{ij} w_{ij}^k) .$$

Our subproblem becomes one of choosing that cycle in the network of ship k for which (13) is a minimum. Since $a_{kg}^{ij} = w_{ij}^k$ is the ship's capacity, if the arc (i,j) is used in the cycle g and zero otherwise, the sum in (13) is simply the sum of the ship capacities on arcs (i,j) weighted by p_{ij} and the times weighted by q_k around the cycle g. Note that t_{kg} is the sum of times on arcs (i,j) around the cycle. Unfortunately the problem in this form is that of finding a most negative cycle in a graph whose arc distances are given. This class of problems includes as a special case the difficult travelling salesman problem.

We got around this difficulty by a change of uni set $x_{kg} = \bar{x}_{kg}/t_{kg}$. The relative cost coefficients for the new problem become

$$(14) \qquad -q_k - (\sum_g p_{ij} w^k_{ij}/t_{kg})$$

where g denotes the $(i,j) \in$ cycle g

Since q_k for fixed k is constant and $t_{kg} = \sum_g t^k_{ij}$ the subproblem becomes one of finding that cycle g^* that minimizes the ratio

$$(15) \qquad (-\sum_g p_{ij} c^k_{ij})/(\sum_g t^k_{ij})$$

which fortunately, as we have seen, is a solveable problem!

REFERENCES

[1] Dantzig, G. B., Blattner, W., and Rao, M. R., "All Shortest Routes from a Fixed Origin in a Graph," Research Report No. 66-2, Operations Research House, Stanford University, November 1966.

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13. ABSTRACT

Associated with each arc of a directed graph are two numbers which will be called arc cost and arc time. The problem is to find that cycle in a graph whose ratio of the sum of arc costs to the sum of arc times around the cycle is a minimum. Application of the simplex method reduces to finding a negative cycle in a graph when arc distances are given which may be positive or negative. The algorithm thus may be viewed as a variant of the shortest route algorithm that yields either the shortest route from a starting node to all others or uncovers a negative cycle.

This problem arose as a sub-problem of a larger problem in which the decomposition principle was used. The application was to ship routing where there are quantities to be shipped between various parts, and vessels have capacities that depend on the type of vessel and type of arc.

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